HEAT TRANSFER IN LAMINAR FLOW. IV.*

EFFECT OF TEMPERATURE-DEPENDENT VISCOSITY

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The Grätz-Nusselt problem solved by the method of the thermal boundary layer has been extended to the case of heat transfer in a liquid with temperature-dependent viscosity. A solution has been found for nonisoviscous flow in a pipe. Appropriateness of the approximate procedure has been verified by confronting it with existing numerical solution. The method has been further applied to the case of a flat duct with heat transfer across one of its walls. The results have been correlated by the relation of the Sieder-Tate type $\text{Nu}_{\text{noniso}} = \text{Nu}_{\text{iso}}(\eta_M/\eta_W)^\alpha$. The values of the coefficient α have been tabulated for different situations.

Heat transfer by forced convection during laminar flow is interesting from the practical point of view mainly in liquids with higher viscosity coefficients where turbulence cannot be produced economically and where natural convection is usually excluded. High viscosity is encountered frequently in substances which do not form any crystalline solid phase -- *e.g.* in melts and concentrated solutions of polymers, oils, tars, glasses *etc.* Simultaneously, a pronounced dependence of viscosity on temperature is characteristic for such substances.

Dependence of the Viscosity on the Temperature

This dependence may be usually correlated by the relation

$$\eta = \eta_0 \exp\left[-AT\right].\qquad(1)$$

For non-Newtonian liquids characterized by the viscosity function in the form

$$\eta = \eta[\tau, T]\qquad(2)$$

it is useful to express A as

$$A \equiv -\partial \ln \eta[\tau, T]/\partial T.\qquad(3)$$

Parameter A is usually greater than $0.01 \ \mathrm{deg}^{-1}$, in highly viscous liquids it assumes normally values of $0.1 \ \mathrm{deg}^{-1}$ and we must not be surprised by values greater by several orders. Low values of A (sometimes even below $0.01 \ \mathrm{deg}^{-1}$) are found only exceptionally with highly consistent liquids, predominantly, however, with concentrated suspensions.

   In contrast to other correlations of the temperature dependence of viscosity, relation (1) brings only one new dimensionless criterion into the problem of simultaneous heat and momentum transfer

$$\psi \equiv A \cdot \Delta T, \tag{4}$$

where ΔT is a characteristic quantity with the dimension of temperature. If we choose for ΔT the maximum temperature difference $T_w - T_0$ in the system, the criterion ψ describes the extent to which the velocity profile is affected by changes in the temperature. At $|\psi| < 0.1$, i.e. where dependence (1) could be linearized, viscosity changes exhibit no appreciable effects. Consequently, the contribution of works using the linearized relation (1) is too limited. At values of $|\psi| > 0.5$ (e.g. with melts of polymers or viscous oils even at temperature differences of 10 deg), nonisoviscous conditions (so-called nonisothermal flow) must be taken into account.

Equations for Nonisoviscous Flow

The viscosity coefficients and consequently the velocity profiles change under the influence of different temperatures in different places inside the pipe. If we limit ourselves to exchangers of usual lengths $X \gg |\psi| R \cdot \mathrm{Pe}^{-1/2}$, an approximation may be accepted for their most significant part, namely that the pressure is a function of x only and that the velocity profile at a given x is obtained by solving the differential equation

$$\frac{\mathrm{d}v_x}{\mathrm{d}r} = \frac{r}{2\eta(r)} \frac{\mathrm{d}p}{\mathrm{d}x} \tag{5}$$

with conditions

$$v_x = 0 \quad \text{for} \quad r = R, \quad 2\pi \int_0^R v_x(r) \, r \, \mathrm{d}r = \pi R^2 U \; ; \tag{6, 7}$$

$\eta(r)$ is the viscosity coefficient corresponding to temperatures in the given section $x = \text{constant}$. Formally, the solution is found as

$$v_x(r, x) = \frac{UR^2 \int_r^R r/(2\eta(r)) \, \mathrm{d}r}{\int_0^R \int_\xi^R r/\eta(r) \, \mathrm{d}r\xi \, \mathrm{d}\xi} \tag{8}$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{UR^2}{\displaystyle\int_0^R \int_\xi^R r/\eta(r)\,\mathrm{d}r\xi\,\mathrm{d}\xi} \, . \tag{9}$$

Heat Transfer in the Thermal Entry Region

In a previous part[1] of this series we have assumed that the radial temperature profile may be approximated in the range $0 < z \leqq Z_e$ (thermal entry region) for a given $z = z_1$ by two functions: for $0 \leqq y \leqq y_0(z_1)$ it is the first eigenfunction of system $(10)-(12)$

$$\frac{\mathrm{d}^2 t_i}{\mathrm{d}y^2} - \frac{1}{1-y}\frac{\mathrm{d}t_i}{\mathrm{d}y} + \beta_i^2 w(y)\,t_i = 0, \quad t_i = 0 \quad \text{for} \quad y = 0, \tag{10, 11}$$

$$t_i = 1 \quad \text{and} \quad \mathrm{d}t_i/\mathrm{d}y = 0 \quad \text{for} \quad y = y_0(z_1); \tag{12}$$

for $y > y_0(z_1)$ it is $t = 1$. $\tag{13}$

The assumption of approximability of the temperature profile by the first eigen-function is acceptable also here, because the shape of the temperature profile in the case of nonisoviscous flow is determined by the velocity profile, too; the greatest influence is, however, due to the velocity profile in an immediately preceding part of the exchanger. (The validity of this statement may be verified by expressing the temperature profile as a result of heat transfer in a series of elementary exchangers with the help of the superposition principle of fundamental solutions[2]).

The division of the temperature profile into two parts facilitates considerably the parallel solution of equations of heat and momentum transfer. The first eigenfunction of the given Sturm-Liouville problem $(10)-(12)$ in the form $t_1(y/y_0)$ has similar shapes even at considerable different courses of the temperature profile $w(y)$. (This is restricted by the conditions that the function is non-descending on the interval $0 \leqq y/y_0 \leqq 1$ and that it reaches values from zero to unity; its derivative is equal to zero at $y/y_0 = 1$). Therefore it is possible to choose the following iterative scheme; to estimate the temperature profile in the boundary layer for a given y_0 (*e.g.* as $t = 3/2y/y_0 - \frac{1}{2}(y/y_0)^3$) and hence to calculate the corresponding velocity profile. We obtain a corrected value for the temperature for this velocity profile as a first eigenfunction of system $(10)-(12)$ and the whole procedure is repeated until agreement between two successive results becomes satisfactory.

Then, if we know temperature profiles in the boundary layer for so many values of y_0 that the whole course of the dependences of the mean mixing temperature t_M and the temperature gradient at the wall $\partial t/\partial y|_{y=0}$ on y_0 may be determined by an

interpolation, we can find the coordinate z corresponding to the value of y_0 with the help of heat balance

$$\frac{dt_M}{dz} = -2 \left. \frac{\partial t}{\partial y} \right|_{y=0} . \tag{14}$$

After inserting $\vartheta = (1 - t_M)^{3/2}$, relation (14) goes over to the formula

$$z(t_M) = \tfrac{1}{3} \int_0^{\vartheta(t_M)} \frac{1}{\left. \dfrac{\partial t}{\partial y} \right|_{y=0,\vartheta}} \frac{1}{\vartheta^{1/3}} \, d\vartheta \tag{15}$$

which is especially convenient for graphical integration due to the low variability of the integrand with respect to the variable of integration.

Solution in the Region of a Developed Temperature Profile

Similarly as in the thermal entry region (at $z < Z_e$, where $y_0 < 1$), we can assume at high values of $z > Z_e$ that the radial temperature profile is affected predominantly by the local velocity profile and that temperature profiles are similar for the same velocity profile. Hence it follows that at $z > Z_0$ and the dimensionless temperature in the pipe axis simultaneously equal to $t(1)$, the velocity profile is the same as for $\psi^+ = \psi t(1)$, $y_0 = 1$, $z = Z_e(\psi^+)$ and the temperature profile is equal to $t(y) = = 1/|\psi| \cdot t_1^+(y)$, where t_1^+ is the first eigenfunction at $\psi = \psi^+$ and $y_0 = 1$.

Numerical results for non-Newtonian flow through a pipe may be linearized to yield the relations

$$Z_e = 0{\cdot}117 , \quad t_M^+ = 0{\cdot}536 - 0{\cdot}036\psi , \quad f(x) = -0{\cdot}238 - 0{\cdot}032x . \tag{16}-(18)$$

The symbol t_M^+ stands for the value of t_M at $z = Z_e$ and the derivative (dz/dt_M) is taken at the same conditions. The function $f(x)$ relates the argument $x = t_M^+ \cdot \psi^+$ to the functional value equal to the product $(dz/dt_M)^+ \cdot t_M$ and, consequently, the same relation holds (in agreement with the above mentioned similarities of the profiles and with relation (14)) between the argument $t_M \cdot \psi$ and the functional value (dz/dt_M). $\cdot t_M$. If we use approximate formulas $(16)-(18)$, we obtain by integration

$$z - 0{\cdot}117 - 0{\cdot}238 \ln \frac{t_M}{0{\cdot}536 - 0{\cdot}036\psi} + 0{\cdot}032\psi [t_M - 0{\cdot}536 + 0{\cdot}036\psi] = 0 . \tag{19}$$

With respect to several approximations made during the calculation of formulas $(16)-(18)$, it is suitable to adjust the numerical constants so that at $\psi = 0$ and $z \to \infty$

relation (19) may pass to the exact solution[3] $t_M = t_{M1} \exp(-b_1^2 z)$ and $Z_e = \ln c_1/b_1^2$ in agreement with the paper[1]; then it holds

$$z - \frac{1}{b_1^2} \ln \frac{t_M}{t_{M1}(1 - a_1\psi)} + a_2\psi \left[t_M - \frac{t_{M1}}{c_1}(1 - a_1\psi) \right] = 0 , \qquad (20)$$
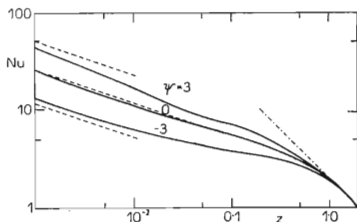

Fig. 1

Mean Nusselt Number Nu for Newtonian Nonisoviscous Flow through a Pipe Exchanger with a Constant Temperature of Its Wall in Dependence on the Dimensionless Length of the Exchanger z and on the Nonisoviscousness Criterion ψ

———— Solution by the method of the boundary layer, ----- course in the limit of $z \to 0$, —·—·— course in the limit of $z \to \infty$.
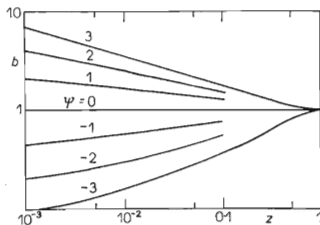

Fig. 2

Dependence of the Dimensionless Local Pressure Gradient $b = (\mathrm{d}p/\mathrm{d}x) \, R/8U\eta_W$ on the Dimensionless Distance from the Beginning of the Pipe Exchanger z and on the Nonisoviscousness Criterion ψ

Solution by the method of the boundary layer.

where the parameters b_1^2, t_{M1}, c_1 follow from the solution for isoviscous heat transfer[9] and $a_1 = 0.067$, $a_2 = 0.032$ for non-Newtonian nonisoviscous flow through the pipe.

Generalized Correlations for Heat Transfer

Results for the heat transfer into a Newtonian liquid with temperature-dependent viscosity (1) may be e.g. formulated as the dependence of the mean Nusselt number based on the total heat transfer and on the arithmetic-mean difference in temperatures

$$\text{Nu} = \frac{2(1 + \varkappa)}{z} \frac{1 - t_M}{1 + t_M} \tag{21}$$

on z (Fig. 1) and as the dependence of local values of the dimensionless loss in pressure b on z (Fig. 2).

These dependences were compared with results obtained by the method of finite differences by other authors[4−6]. Quantitative agreement with the results from the numerical solution was observed; discernible deviations may be found only at lower values of z, i.e. where the thickness of the boundary layer is comparable or even smaller than a step in the radial division of the grid of the computational procedure and, consequently, where the method of finite differences fails, whereas the solution of the boundary layer remains reliable.

Thus, agreement with the numerical solution justifies the approximate mathematical procedure based on the variational method for finding the first eigenfunction in the form of an odd polynomial of the fifth degree and the approximation of the viscosity
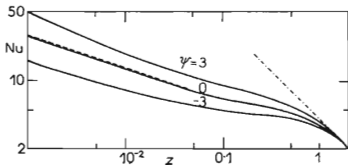

Fig. 3

Dependence of the Mean Nusselt Number Nu for Newtonian Nonisoviscous Flow through a Flat Duct with One Wall Kept at a Constant Temperature and the Other Insulated, on the Dimensionless Length of the Exchanger z and on the Nonisoviscousness Criterion ψ

——— Solution by the method of the boundary layer, ----- course in the limit of $z \to 0$, —·—·—· course in the limit of $z \to \infty$.

profile by a polynomial, which has reduced the whole problem to repeated solution of a quadratic equation. This approximation does not guarantee the convergence only if a pronounced inflection is present on the velocity profile (at $\psi < -3$).

A scheme similar to that described in this work may be applied to other geometric arrangements. There are no fundamental differences in the whole computational procedure for a flact duct with a temperature step on one wall and with the other one insulated (except for the necessity of finding the coordinate of the maximum velocity). Analogous results have been obtained[7] for this case; e.g. relation (20) with values of $a_1 = 0.034$, $a_2 = 0.039$ and the dependence of the Nusselt number on z as depicted on Fig. 3.

Practically it is most usual to correlate results of nonisoviscous heat transfer by comparing the actual arithmetic-mean Nusselt numbers (index noniso) with those calculated for the case of temperature-independent viscosity (index iso) by the relation

$$\mathrm{Nu}_{\mathrm{noniso}} = \mathrm{Nu}_{\mathrm{iso}}(\eta_{\mathrm{M}}/\eta_{\mathrm{W}})^{\alpha}. \qquad (22)$$

The value of α is often set equal to 0.14 as determined by Sieder and Tate[8] during experiments with both laminar and turbulent heat transfer in a pipe. Quite generally, α is a function of both z and ψ, and, for non-Newtonian liquids, of their rheologic behaviour. The value of α has turned out to be practically independent of ψ in the usually sufficient range $-3 \leq \psi \leq 3$ and only slightly variable with the dimensionless length of the exchanger z (Table I). It follows from Christiansen's results[9] that the dependence of α on the rheologic behaviour is insignificant as well and that in formula (22) it is possible to employ the same α as for Newtonian liquids.

TABLE I
Mean Values of Exponent α for a Newtonian Liquid, $-3 \leq \psi \leq 3$

z	0	0.001	0.01	0.1	∞
α pipe	0.25	0.19	0.17	0.16	0.144
α flat duct with asymmetric heat transfer	—	0.20	0.16	0.14	0.144

The use of classical correlation (22) for engineering calculations of heat transfer during nonisoviscous laminar flow in a pipe and in an asymmetrically heated flat duct is justified by the calculated results; for the exponent, it is however necessary to take somewhat higher value than the usually employed $\alpha = 0.14$.

LIST OF SYMBOLS

A	material parameter; see (1) and (3) $(\deg^{-1})$
a_1, a_2	constants in relation (20)
b	$= (dp/dx)\,R^2/(8 U \eta_W)$ dimensionless local pressure gradient
b_1^2	first eigenvalue of the equation for isoviscous heat transfer[10]
c_1	first coefficient of the solution for the isoviscous temperature profile[10]
c_p	specific heat $(\text{cal g}^{-1}\,\text{deg}^{-1})$
k	thermal conductivity $(\text{cal cm}^{-1}\,\text{s}^{-1}\,\text{deg}^{-1})$
dp/dx	local pressure gradient $(\text{g cm}^{-2}\,\text{s}^{-2})$
r	radial coordinate (cm)
R	radius of the pipe (width of the flat duct) (cm)
T	temperature (°C)
T_0	temperature of the liquid entering the exchanger (°C)
T_W	temperature of the exchanger wall (°C)
T_M	mean mixing temperature of the liquid flowing through the given section (°C)
t	$= (T - T_W)/(T_0 - T_W)$ dimensionless temperature
t_1	first eigenfunction of system (10)—(12)
t_M	dimensionless mean mixing temperature
t_{M1}	coefficient of the solution for isoviscous heat transfer[1]
U	mean velocity of the liquid (cm s^{-1})
v_x	axial velocity (cm s^{-1})
w	$= v_x/U$, dimensionless velocity
x	distance from the beginning of the exchanger (cm)
X	length of the exchanger (cm)
y	$= (1 - r)/R$, dimensionless distance from the wall
y_0	dimensionless width of the thermal boundary layer
z	$= 2x/(R\,\text{Pe})$, dimensionless axial coordinate
Z_e	dimensionless length of the thermal entry region
z_1	chosen value for $z < Z_e$
Nu	Nusselt number based on the arithmetic-mean temperature difference (21)
Nu_{iso}	value of Nu for isoviscous heat transfer
Nu_{noniso}	actual value of Nu
Pe	$= 2RU\varrho c_p/k$, Péclet number
α	exponent of Sieder-Tate correction (22)
β_1^2	eigenvalue of system (10)—(12)
η	viscosity $(\text{g cm}^{-1}\,\text{s}^{-1})$
η_M	viscosity at temperature T_M $(\text{g cm}^{-1}\,\text{s}^{-1})$
η_W	viscosity at temperature T_W $(\text{g cm}^{-1}\,\text{s}^{-1})$
η_0	material parameter (1) $(\text{g cm}^{-1}\,\text{s}^{-1})$
ϑ	transformed variable (14)
$\varkappa$	geometric simplex equal to zero for a pipe and unity for a flat duct
ϱ	specific weight (g cm^{-3})
τ	tangential stress $(\text{g cm}^{-1}\,\text{s}^{-2})$
ψ	criterion of nonisoviscousness of the flow
ψ^+	$= \psi \cdot t(1)$

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